Integration by Substitution

(4.7) The evaluation of certain integrals becomes easy if we change the variable of integration by some suitable substitution. Suppose, we have to evaluate

$$\int f(x)\,dx.$$

We make the substitution $x = g(z)$ to change the variable x into z. Then

$$dx = g'(z)\,dz$$

Hence $$\int f(x)\,dx = \int f(g(z))\,g'(z)\,dz.$$

The substitution $x = g(z)$ is to be such that the transformed integral on the right-hand side of the above equation is easier to evaluate than the given integral. No specific rule can be given for selecting the substitution. If, after a substitution, the new integral becomes more complicated, then either some other substitution should be tried or other methods need to be employed.

A justification for the substitution method follows from the chain rule. We know that

$$\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$$

Therefore, $$\int f'(g(x))\,g'(x)\,dx = f(g(x)).$$

New let $z = g(x)$, so that $dz = g'(x)\,dx$

and we have $$\int f'(g(x))\,g'(x)\,dx = \int f'(z)\,dz = f(z).$$

Example 1. Evaluate $\int \frac{e^{\arctan x}}{1+x^2}\,dx$.

Solution. Put $z = \arctan x$

$$dz = \frac{1}{1+x^2}\,dx$$

$$\int \frac{e^{\arctan x}}{1+x^2}\,dx = \int e^z\,dz = e^z = \boxed{e^{\arctan x}}$$

Example 2. Compute $\int \sqrt{\sin x}\,\cos x\,dx$.

Solution. Let $\sin x = z$, so that $\cos x\,dx = dz$. Substituting, we have the given integral $= \int \sqrt{z}\,dz = \frac{z^{3/2}}{3/2} = \frac{2}{3}(\sin x)^{3/2} = \frac{2}{3}\sqrt{\sin^3 x}$.

Example 3. Evaluate $\int \frac{dx}{1+\sqrt{x+1}}$.

Solution. Let $\sqrt{x+1} = z$ so that

$$z^2 = x+1. \text{ Then}$$

$$2z\,dz = dx \text{ and on substitution, we have}$$

$$\int \frac{dx}{1+\sqrt{x+1}} = \int \frac{2z\,dz}{1+z}$$

$$= \int \left(2 - \frac{2}{1+z}\right) dz$$

$$= 2z - 2\ln(1+z)$$

$$= 2\sqrt{x+1} - 2\ln\left(1+\sqrt{x+1}\right).$$

Example 4. Compute $\int \frac{\cos^6 2\theta}{\sin^8 2\theta}\, d\theta$.

Solution. Let $\cot 2\theta = z$ so that

$$-2\csc^2 2\theta\, d\theta = dz \text{ and so on substitution, we have}$$

$$\int \frac{\cos^6 2\theta}{\sin^8 2\theta}\, d\theta = \int \cot^6 2\theta \csc^2 2\theta\, d\theta$$

$$= -\frac{1}{2}\int z^6\, dz$$

$$= -\frac{1}{2}\frac{z^7}{7} = -\frac{1}{14}\cot^7 2\theta.$$

Example 5. Evaluate $\int \frac{2x+1}{(x^2+x+1)^{5/2}}\, dx$

Solution. We note that $2x+1$ is the derivative of x^2+x+1 and so

$$\int \frac{2x+1}{(x^2+x+1)^{5/2}}\, dx = \int (x^2+x+1)^{-5/2}\, d(x^2+x+1)$$

$$= \frac{(x^2+x+1)^{-3/2}}{-3/2}$$

$$= -\frac{2}{3}\frac{1}{(x^2+x+1)^{3/2}}$$

Alternatively. Let $x^2 + x + 1 = z$ so that $(2x + 1)\,dx = dz$ and on substitution, we have

$$\int \frac{2x+1}{(x^2+x+1)^{5/2}}\,dx = \int \frac{dz}{z^{5/2}}$$

$$= \int z^{-5/2}\,dz$$

$$= \frac{z^{-3/2}}{-3/2} = \frac{-2}{3z^{3/2}} = \frac{-2}{3} \cdot \frac{1}{(x^2+x+1)^{3/2}}.$$

Example 6. Evaluate $\int \tan^2\theta \sec^4\theta \, d\theta$.

Solution. Let $\tan\theta = z$ so that $\sec^2\theta\, d\theta = dz$ and on substitution, we have

$$\int \tan^2\theta \sec^4\theta\, d\theta = \int z^2 (1 + \tan^2\theta)\sec^2\theta\, d\theta$$

$$= \int z^2 (1 + z^2)\,dz = \frac{z^3}{3} + \frac{z^5}{5} = \boxed{\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}}$$

Exercise Set 4.2

Evaluate (Problems 1 – 22):

1. $\int \frac{dx}{\sqrt{a^2+x^2}}$

2. $\int \frac{dx}{\sqrt{x^2-a^2}}$

3. $\int \tan x\, dx$

4. $\int \cot x\, dx$

5. $\int \sec x\, dx$

6. $\int \csc x\, dx$

7. $\int (ax^2 + 2bx + c)^n (ax + b)\, dx$

8. $\int \sqrt{\frac{1+x}{1-x}}\, dx$

9. $\int \frac{dx}{a + \sqrt{bx+c}}$

10. $\int \frac{dx}{(1+x^2)\arctan x}$

11. $\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx$

12. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

13. $\int \sqrt{e^{2x} + e^{3x}}\, dx$

14. $\int \frac{dx}{e^x + e^{-x}}$

15. $\int \frac{e^{2x}\,dx}{\sqrt{e^x - 1}}$

16. $\int \frac{\cos(\ln x)}{x}\, dx$

$(x+2)(x^2+3)$

Integration of Irrational Functions

(4.11) The following trigonometric substitutions will be used to handle integrals involving the indicated radicals:

(i) $\sqrt{x^2+a^2}$, put $x = a\tan\theta$, $\quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

(ii) $\sqrt{a^2-x^2}$, put $x = a\sin\theta$, $\quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) $\sqrt{x^2-a^2}$, put $x = a\sec\theta$, $\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$

In case where the above substitutions lead to complicated integrals, it is sometimes convenient to make the **hyperbolic substitutions**.

(iv) $x = a\sinh z$ for integrals involving $\sqrt{x^2+a^2}$

(v) $x = a\cosh z$ for integrals involving $\sqrt{x^2-a^2}$.

Example 24. Evaluate $I = \int \frac{dx}{x^2\sqrt{9-x^2}}$.

Solution. Put $x = 3\sin\theta$, $\quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

or $\quad dx = 3\cos\theta\, d\theta$

$$I = \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta \cdot 3\cos\theta}$$

$$= \frac{1}{9}\int \csc^2\theta \cdot d\theta$$

$$= \frac{-1}{9}\cot\theta$$

$$= \frac{-1}{9} \cdot \frac{\sqrt{9-x^2}}{x}.$$

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